

**THE INTEGRALS IN GRADSHTEYN AND RYZHIK.
PART 6: THE BETA FUNCTION.**

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ABSTRACT. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Ryzhik that can be reduced to the beta function.

1. INTRODUCTION

The table of integrals [2] contains some evaluations that can be derived by elementary means from the *beta function*, defined by

$$(1.1) \quad B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx.$$

The convergence of the integral in (1.1) requires $a, b > 0$. This definition appears as **3.191.3** in [2].

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Ryzhik [2], that involve this function. In this part, we restrict to algebraic integrands leaving the trigonometric forms for a future publication. This paper complements [3] that dealt with the *gamma function* defined by

$$(1.2) \quad \Gamma(a) := \int_0^\infty x^{a-1} e^{-x} dx.$$

These functions are related by the functional equation

$$(1.3) \quad B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.$$

A proof of this identity can be found in [1].

The special values $\Gamma(n) = (n-1)!$ and

$$(1.4) \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!}$$

for $n \in \mathbb{N}$, will be used to simplify the values of the integrals presented here. Proofs of these formulas can be found in [3] as well as in Proposition 2.1 below.

The other property that will be employed frequently is

$$(1.5) \quad \Gamma(a) \Gamma(1-a) = \frac{\pi}{\sin \pi a}.$$

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The reader will find in [1] a proof based on the product representation of these functions. A challenging problem is to produce a proof that only employs changes of variables.

The table [2] contains some direct values:

$$(1.6) \quad \int_0^1 \frac{x^p dx}{(1-x)^p} = \frac{p\pi}{\sin p\pi}$$

is **3.192.1** and is evaluated by identifying it as $B(p+1, 1-p)$. Formula **3.192.2** is

$$(1.7) \quad \int_0^1 \frac{x^p dx}{(1-x)^{p+1}} = -\frac{\pi}{\sin p\pi}$$

has the value $B(p+1, -p) = \Gamma(p+1)\Gamma(-p)$. Next, **3.192.3** is

$$(1.8) \quad \int_0^1 \frac{(1-x)^p}{x^{p+1}} dx = -\frac{\pi}{\sin p\pi}$$

and the change of variables $t = 1/x$ in **3.192.4** produces

$$(1.9) \quad \int_1^\infty (x-1)^{p-1/2} \frac{dx}{x} = \int_0^1 t^{-p-1/2} (1-t)^{p-1/2} dt$$

and this is

$$(1.10) \quad B\left(\frac{1}{2}-p, \frac{1}{2}+p\right) = \Gamma\left(\frac{1}{2}-p\right) \Gamma\left(\frac{1}{2}+p\right) = \frac{\pi}{\cos p\pi},$$

as stated in [2].

Let $b = \frac{1}{2}$ in (1.1) to obtain

$$(1.11) \quad \int_0^1 \frac{x^{a-1} dx}{\sqrt{1-x}} = B\left(a, \frac{1}{2}\right) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)}.$$

The special values $a = n+1$ and $a = n + \frac{1}{2}$ appear as **3.226.1** and **3.226.2**, respectively.

2. ELEMENTARY PROPERTIES

Many of the properties of the beta function can be established by simple changes of variables. For example, letting $y = 1-x$ in (1.1) yields the symmetry

$$(2.1) \quad B(a, b) = B(b, a).$$

It should not be surprising that a clever change of variables might lead to a beautiful result. This is illustrated following Serret [4]. Start with

$$\begin{aligned} B(a, a) &= \int_0^1 (x-x^2)^{a-1} dx \\ &= 2 \int_0^{1/2} \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2 \right]^{a-1} dx. \end{aligned}$$

The natural change of variables $v = \frac{1}{2} - x$ yields

$$(2.2) \quad B(a, a) = 2 \int_0^{1/2} \left(\frac{1}{4} - v^2 \right)^{a-1} dv.$$

The next step is now clear: let $s = 4v^2$ to produce

$$(2.3) \quad B(a, a) = 2^{1-2a} B\left(a, \frac{1}{2}\right).$$

The functional equation (1.3) converts this identity into Legendre's original form:

Proposition 2.1. The gamma function satisfies

$$(2.4) \quad \Gamma\left(a + \frac{1}{2}\right) = \frac{\Gamma(2a) \Gamma(\frac{1}{2})}{\Gamma(a) 2^{2a-1}}.$$

In particular, for $a = n \in \mathbb{N}$, this yields (1.4).

3. ELEMENTARY CHANGES OF VARIABLES

The integral (1.1) defining the beta function can be transformed by changes of variables. For example, the new variable $x = t/u$, reduces (1.1) to

$$(3.1) \quad \int_0^u t^{a-1} (u-t)^{b-1} dt = u^{a+b-1} B(a, b),$$

that appears as **3.191.1** in [2]. The effect of this change of variables is to express the beta function as an integral over a finite interval. Observe that the integrand vanishes at both end points. Similarly, the change $t = (v-u)x + u$ maps the interval $[0, 1]$ to $[u, v]$. It yields

$$(3.2) \quad \int_u^v (t-u)^{a-1} (v-t)^{b-1} dt = (v-u)^{a+b-1} B(a, b).$$

This is **3.196.3** in [2]. The special case $u = 0$, $v = n$ and $a = \nu$, $b = n + 1$ appears as **3.193** in [2] as

$$(3.3) \quad \int_0^n x^{\nu-1} (n-x)^n dx = \frac{n^{\nu+n} n!}{\nu(\nu+1)(\nu+2) \cdots (\nu+n)}.$$

Several integrals in [2] can be obtained by a small variation of the definition. For example, the integral

$$(3.4) \quad \int_0^1 (1-x^a)^{b-1} dx = \frac{1}{a} B(1/a, b)$$

can be obtained by the change of variables $t = x^a$. This appears as **3.249.7** in [2] and illustrates the fact that it not necessary for the integrand to vanish at *both* end points. The special case $a = 2$ appears as **3.249.5**:

$$(3.5) \quad \int_0^1 (1-x^2)^{b-1} dx = \frac{1}{2} B\left(\frac{1}{2}, b\right) = 2^{2b-2} B(b, b),$$

where the second identity follows from Legendre's duplication formula (2.4).

The change of variables $t = cx$ produces a scaled version:

$$(3.6) \quad \int_0^c (c^a - t^a)^{b-1} dt = \frac{1}{a} c^{a(b-1)+1} B(1/a, b).$$

The special case $a = 2$ yields

$$(3.7) \quad \int_0^c (c^2 - t^2)^{b-1} dt = \frac{c^{2b-1}}{2} B(1/2, b).$$

The choice $b = n + \frac{1}{2}$ appears as **3.249.2** in [2]:

$$(3.8) \quad \int_0^c (c^2 - t^2)^{n-1/2} dt = \frac{\pi c^{2n}}{2^{2n+1}} \binom{2n}{n}.$$

Similarly **3.251.1** in [2] is

$$(3.9) \quad \int_0^1 x^{c-1} (1-x^a)^{b-1} dx = \frac{1}{a} B\left(\frac{c}{a}, b\right).$$

The change of variables $t = 1/x$ converts (1.1) into

$$(3.10) \quad \int_1^\infty t^{-a-b} (t-1)^{b-1} dt = B(a, b).$$

Letting $t = x^p$ yields

$$(3.11) \quad \int_1^\infty x^{p(1-a-b)-1} (x^p - 1)^{b-1} dx = \frac{1}{p} B(a, b).$$

The special case $\nu = b$ and $\mu = p(1-a-b)$ is **3.251.3**:

$$(3.12) \quad \int_1^\infty x^{\mu-1} (x^p - 1)^{\nu-1} dx = \frac{1}{p} B(1-\nu-\mu/p, \nu).$$

4. INTEGRALS OVER A HALF-LINE

The beta function can also be expressed as an integral over a half-line. The change of variables $t = x/(1-x)$ maps $[0, 1]$ onto $[0, \infty)$ and it produces from (1.1)

$$(4.1) \quad B(a, b) = \int_0^\infty \frac{t^{a-1} dt}{(1+t)^{a+b}}.$$

In particular, if $a+b=1$, using (1.3) and (1.5), we obtain

$$(4.2) \quad \int_0^\infty \frac{t^{a-1} dt}{1+t} = \frac{\pi}{\sin \pi a}.$$

This can be scaled to produce, for $a > 0$ and $c > 0$,

$$(4.3) \quad \int_0^\infty \frac{x^{a-1} dx}{x+c} = \frac{\pi}{\sin \pi a} c^{a-1} \quad \text{for } c > 0$$

that appears as **3.222.2** in [2]. In the case $c < 0$ we have a singular integral. Define $b = -c > 0$ and $s = x/b$, so now we have to evaluate

$$(4.4) \quad I = -b^{a-1} \int_0^\infty \frac{s^{a-1} ds}{1-s}.$$

The integral is considered as a Cauchy principal value

$$(4.5) \quad I = \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{s^{a-1} ds}{(1-s)^{1-\epsilon}} + \int_1^\infty \frac{s^{a-1} ds}{(1-s)^{1-\epsilon}}.$$

Let $y = 1/s$ in the second integral and evaluate them in terms of the beta function to produce

$$(4.6) \quad I = \lim_{\epsilon \rightarrow 0} \epsilon \Gamma(\epsilon) \times \frac{1}{\epsilon} \left(\frac{\Gamma(a)}{\Gamma(a+\epsilon)} - \frac{\Gamma(1-a-\epsilon)}{\Gamma(1-a)} \right).$$

Use L'Hopital's rule to evaluate and obtain

$$(4.7) \quad I = -\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(1-a)}{\Gamma(a)}.$$

Using the relation $\Gamma(a)\Gamma(1-a) = \pi \operatorname{cosec} \pi a$, this reduces to $\pi \cot \pi a$. Therefore we have

$$(4.8) \quad \int_0^\infty \frac{x^{a-1} dx}{x+c} = -\frac{\pi}{\tan \pi a} (-c)^{a-1} \quad \text{for } c < 0$$

The change of variables $x = e^{-t}$ produces, for $c < 0$,

$$(4.9) \quad \int_{-\infty}^\infty \frac{e^{-\mu t} dt}{e^{-t} + c} = -\pi \cot(\mu\pi) (-c)^{\mu-1}.$$

The special case $c = -1$ appears as **3.313.1**:

$$(4.10) \quad \int_{-\infty}^\infty \frac{e^{-\mu t} dt}{1 - e^{-t}} = \pi \cot(\mu\pi).$$

We now consider several examples in [2] that are direct consequences of (4.3) and (4.8). In the first example, we combine (4.3) with the partial fraction decomposition

$$(4.11) \quad \frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right)$$

leads to **3.223.1**:

$$(4.12) \quad \int_0^\infty \frac{x^{\mu-1} dx}{(x+b)(x+a)} = \frac{\pi}{b-a} (a^{\mu-1} - b^{\mu-1}) \operatorname{cosec}(\pi\mu).$$

Similarly,

$$(4.13) \quad \frac{1}{x+b} - \frac{1}{x-a} = \frac{a+b}{(a-x)(b+x)}$$

leads to **3.223.2**:

$$(4.14) \quad \int_0^\infty \frac{x^{\mu-1} dx}{(b+x)(a-x)} = \frac{\pi}{a+b} (b^{\mu-1} \operatorname{cosec}(\mu\pi) + a^{\mu-1} \cot(\mu\pi)),$$

using (4.3) and (4.8). The result **3.223.3**:

$$(4.15) \quad \int_0^\infty \frac{x^{\mu-1} dx}{(a-x)(b-x)} = \pi \cot(\mu\pi) \frac{a^{\mu-1} - b^{\mu-1}}{b-a},$$

follows from

$$(4.16) \quad \frac{1}{(a-x)(b-x)} = \frac{1}{a-b} \left(\frac{1}{b-x} - \frac{1}{a-x} \right).$$

Finally, **3.224**:

$$(4.17) \quad \int_0^\infty \frac{(x+b)x^{\mu-1} dx}{(x+a)(x+c)} = \frac{\pi}{\sin(\mu\pi)} \left(\frac{a-b}{a-c} a^{\mu-1} + \frac{c-b}{c-a} c^{\mu-1} \right),$$

follows from

$$(4.18) \quad \frac{x+b}{(x+a)(x+c)} = \frac{b-a}{c-a} \frac{1}{x+a} - \frac{b-c}{c-a} \frac{1}{x+c}.$$

We can now transform (4.1) to the interval $[0, 1]$ by splitting $[0, \infty)$ as $[0, 1]$ followed by $[1, \infty)$. In the second integral, we let $t = 1/s$. The final result is

$$(4.19) \quad B(a, b) = \int_0^1 \frac{t^{a-1} + t^{b-1}}{(1+t)^{a+b}} dt.$$

This formula, that appears as **3.216.1**, makes it apparent that the beta function is symmetric: $B(a, b) = B(b, a)$. The change of variables $s = 1/t$ converts (4.19) into **3.216.2**:

$$(4.20) \quad B(a, b) = \int_1^\infty \frac{s^{a-1} + s^{b-1}}{(1+s)^{a+b}} ds.$$

It is easy to introduce a parameter: let $c > 0$ and consider the change of variables $t = cx$ in (4.1) to obtain

$$(4.21) \quad \int_0^\infty \frac{x^{a-1} dx}{(1+cx)^{a+b}} = c^{-a} B(a, b),$$

that appears as **3.194.3**. We can now shift the lower limit of integration via $t = x+u$ to produce

$$(4.22) \quad \int_u^\infty (t-u)^{a-1} (t+v)^{-a-b} dt = (u+v)^{-b} B(a, b),$$

where $v = 1/c - u$. This is **3.196.2**, where v is denoted by β . Now let $b = c - a$ in the special case $v = 0$ to obtain

$$(4.23) \quad \int_u^\infty (t-u)^{a-1} t^{-c} dt = u^{a-c} B(a, c-a).$$

This appears as **3.191.2**.

We now write (4.1) using the change of variables $t = x^c$. It produces

$$(4.24) \quad \int_0^\infty \frac{x^{ac-1} dx}{(1+x^c)^{a+b}} = \frac{1}{c} B(a, b).$$

The special case $c = 2$ and $a = 1 + \mu/2$, $b = 1 - \mu/2$ produces **3.251.6** in the form

$$(4.25) \quad \int_0^\infty \frac{x^{\mu+1} dx}{(1+x^2)^2} = \frac{\mu\pi}{4 \sin \mu\pi/2}.$$

Now let $b = 1 - a$ and choose $a = p/c$ to obtain

$$(4.26) \quad \int_0^\infty \frac{x^{p-1} dx}{1+x^c} = \frac{1}{c} B\left(\frac{p}{c}, \frac{c-p}{c}\right) = \frac{\pi}{c} \operatorname{cosec}(\pi p/c).$$

This appears as **3.241.2** in [2].

Similar arguments establish **3.196.4**:

$$(4.27) \quad \int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b} \operatorname{cosec}(\nu\pi) \left(\frac{b}{b-a}\right)^\nu.$$

Indeed, the change of variables $t = x - 1$ yields

$$(4.28) \quad \int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = \int_0^\infty \frac{dt}{[(a-b)-bt] t^\nu},$$

and scaling via the new variable $z = bt/(b-a)$ gives

$$(4.29) \quad \int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{1}{b} \left(\frac{b}{b-a}\right)^\nu \int_0^\infty \frac{dz}{(1+z) z^\nu}.$$

The result follows from (4.1) and the value

$$(4.30) \quad B(\nu, 1-\nu) = \Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}.$$

The same argument gives **3.196.5**:

$$(4.31) \quad \int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b} \operatorname{cosec}(\nu\pi) \left(\frac{b}{a-b} \right)^\nu.$$

5. SOME DIRECT EVALUATIONS

There are many more integrals in [2] that can be evaluated in terms of the beta function. For example, **3.221.1** states that

$$(5.1) \quad \int_a^\infty \frac{(x-a)^{p-1} dx}{x-b} = \pi(a-b)^{p-1} \operatorname{cosec} \pi p.$$

To establish these identities, we assume that $a > b$ to avoid the singularities. The change of variables $t = (x-a)/(a-b)$ yields

$$(5.2) \quad \int_a^\infty \frac{(x-a)^{p-1} dx}{x-b} = (a-b)^{p-1} \int_0^\infty \frac{t^{p-1} dt}{1+t},$$

and this integral appears in (4.2).

Similarly, **3.221.2** states that

$$(5.3) \quad \int_{-\infty}^a \frac{(a-x)^{p-1} dx}{x-b} = -\pi(b-a)^{p-1} \operatorname{cosec} \pi p.$$

This is evaluated by the change of variables $y = (a-x)/(b-a)$.

The table contains several evaluations that are elementary corollaries of (4.1). Starting with

$$(5.4) \quad \int_0^\infty \frac{x^a dx}{(1+x)^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1) \Gamma(b-a-1)}{\Gamma(b)},$$

we find the case $a = p$ and $b = 3$ in **3.225.3**:

$$(5.5) \quad \int_0^\infty \frac{x^p dx}{(1+x)^3} = \frac{\Gamma(p+1) \Gamma(2-p)}{\Gamma(3)} = \frac{p(1-p)}{2} \frac{\pi}{\sin(p\pi)},$$

using elementary properties of the gamma function.

The change of variables $t = 1+x$ converts (5.4) into

$$(5.6) \quad \int_1^\infty \frac{(t-1)^a dt}{t^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1) \Gamma(b-a-1)}{\Gamma(b)}.$$

The special case $a = p-1$ and $b = 2$ gives

$$(5.7) \quad \int_1^\infty \frac{(t-1)^{p-1} dt}{t^2} = \Gamma(p) \Gamma(2-p) = (1-p) \Gamma(p) \Gamma(1-p) = \frac{\pi(1-p)}{\sin(p\pi)}.$$

This appears as **3.225.1**. Similarly, the case $a = 1-p$ and $b = 3$ produces **3.225.2**:

$$(5.8) \quad \int_1^\infty \frac{(t-1)^{1-p} dt}{t^3} = \frac{\Gamma(2-p) \Gamma(1+p)}{\Gamma(3)} = \frac{1}{2} p(1-p) \Gamma(p) \Gamma(1-p) = \frac{\pi p(1-p)}{2 \sin(p\pi)}.$$

6. INTRODUCING PARAMETERS

It is often convenient to introduce free parameters in a definite integral. Starting with (4.1), the change of variables $t = \frac{u}{v}x^c$ yields

$$(6.1) \quad B(a, b) = cu^av^b \int_0^\infty \frac{t^{ac-1} dt}{(v + ut^c)^{a+b}}.$$

This formula appears as **3.241.4** in [2] with the parameters

$$(6.2) \quad a = \frac{\mu}{\nu}, b = n + 1 - \frac{\mu}{\nu}, c = \nu, u = q, \text{ and } v = p,$$

in the form

$$\int_0^\infty \frac{x^{\mu-1} dx}{(p + qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu) \Gamma(n+1 - \mu/\nu)}{\Gamma(n+1)}.$$

This is a messy notation and it leaves the wrong impression that n should be an integer.

- The special case $v = c = 1$ and $b = p + 1 - a$ produces

$$(6.3) \quad \int_0^\infty \frac{t^{a-1} dt}{(1 + ut)^{p+1}} = \frac{1}{u^a} B(a, p + 1 - a).$$

This appears as **3.194.4** in [2], except that it is written in terms of binomial coefficients as

$$(6.4) \quad \int_0^\infty \frac{t^{a-1} dt}{(1 + ut)^{p+1}} = (-1)^p \frac{\pi}{u^a} \binom{a-1}{p} \operatorname{cosec}(\pi a).$$

We prefer the notation in (6.3).

- The special case $v = c = 1$ and $b = 2 - a$ produces

$$(6.5) \quad \int_0^\infty \frac{t^{a-1} dt}{(1 + ut)^2} = \frac{1}{u^a} B(a, 2 - a).$$

Using (1.3) and (1.5) yields the form

$$(6.6) \quad \int_0^\infty \frac{t^{a-1} dt}{(1 + ut)^2} = \frac{(1-a)\pi}{u^a \sin \pi a}.$$

This appears as **3.194.6** in [2].

- The special case $u = v = 1$ and $c = q$, and choosing $a = p/q$ and $b = 2 - p/q$ yields **3.241.5** in the form

$$(6.7) \quad \int_0^\infty \frac{x^{p-1} dx}{(1 + x^q)^2} = \frac{q-p}{q^2} \frac{\pi}{\sin(\pi p/q)}.$$

- The special case $c = 1$ and $a = m + 1$, $b = n - m - \frac{1}{2}$ produces

$$(6.8) \quad \int_0^\infty \frac{t^m dt}{(v + ut)^{n+\frac{1}{2}}} = \frac{1}{u^{m+1} v^{n-m-\frac{1}{2}}} B\left(m+1, n-m-\frac{1}{2}\right)$$

Using (1.3) and (1.4) this reduces to

$$(6.9) \quad \int_0^\infty \frac{t^m dt}{(v + ut)^{n+\frac{1}{2}}} = \frac{m! n! (2n-2m-2)!}{(n-m-1)! (2n)!} 2^{2m+2} \frac{v^{m-n+1/2}}{u^{m+1}},$$

for $m, n \in \mathbb{N}$, with $n > m$. This appears as **3.194.7** in [2].

- The special case $u = v = 1$ and $b = \frac{1}{2} - a$ yields

$$(6.10) \quad \int_0^\infty \frac{t^{ac-1} dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(a, \frac{1}{2} - a\right).$$

Writing $a = p/c$ we recover **3.248.1**:

$$(6.11) \quad \int_0^\infty \frac{t^{p-1} dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(\frac{p}{c}, \frac{1}{2} - \frac{p}{c}\right).$$

- Now replace v by v^2 in (6.1). Then, with $u = 1$, $a = \frac{1}{2}$, $c = 2$, so that $ac = 1$ and $b = n - \frac{1}{2}$ we obtain

$$(6.12) \quad \int_0^\infty \frac{dt}{(v^2 + t^2)^n} = \frac{1}{2v^{2n-1}} B\left(\frac{1}{2}, n - \frac{1}{2}\right).$$

This can be written as

$$(6.13) \quad \int_0^\infty \frac{dt}{(v^2 + t^2)^n} = \frac{\sqrt{\pi} \Gamma(n - 1/2)}{2\Gamma(n) v^{2n-1}}$$

that appears as **3.249.1** in [2].

- The special case $v = 1$, $c = 2$ and $b = \frac{n}{2} - a$ in (6.1) yields

$$(6.14) \quad \int_0^\infty \frac{t^{2a-1} dt}{(1 + ut^2)^{n/2}} = \frac{1}{2u^a} B\left(a, \frac{n}{2} - a\right).$$

Now $a = 1/2$ gives

$$(6.15) \quad \int_0^\infty (1 + ut^2)^{-n/2} dt = \frac{1}{2\sqrt{u}} B\left(\frac{1}{2}, \frac{n-1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{u}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(n/2)}.$$

It is curious that the table [2] contains **3.249.8** as the special case $u = 1/(n-1)$ of this evaluation.

- We now put $u = v = 1$ and $c = 2$ in (6.1). Then, with $b = 1 - \nu - a$ and $a = \mu/2$, we obtain **3.251.2**:

$$(6.16) \quad \int_0^\infty \frac{t^{\mu-1} dt}{(1+t^2)^{1-\nu}} = \frac{1}{2} B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right).$$

- We now consider the case $c = 2$ in (6.1):

$$(6.17) \quad \int_0^\infty \frac{t^{2a-1} dt}{(v + ut^2)^{a+b}} = \frac{1}{2u^a v^b} B(a, b).$$

The special case $a = m + \frac{1}{2}$ and $b = n - m + \frac{1}{2}$ yields

$$(6.18) \quad \int_0^\infty \frac{t^{2m} dt}{(v + ut^2)^{n+1}} = \frac{\Gamma(m + 1/2) \Gamma(n - m + 1/2)}{2u^{m+1/2} v^{n-m+1/2} \Gamma(n + 1)},$$

and using (1.4) we obtain **3.251.4**:

$$(6.19) \quad \int_0^\infty \frac{t^{2m} dt}{(v + ut^2)^{n+1}} = \frac{\pi(2m)!(2n-2m)!}{2^{2n+1} m! (n-m)! n! u^{m+1/2} v^{n-m+1/2}},$$

for $n, m \in \mathbb{N}$ with $n > m$.

On the other hand, if we choose $a = m + 1$ and $b = n - m$ we obtain **3.251.5**:

$$(6.20) \quad \int_0^\infty \frac{t^{2m+1} dt}{(v + ut^2)^{n+1}} = \frac{\Gamma(m+1)\Gamma(n-m)}{2u^{m+1}v^{n-m}\Gamma(n+1)} = \frac{m!(n-m-1)!}{2n!u^{m+1}v^{n-m}}.$$

Several evaluation in [2] come from the form

$$(6.21) \quad \int_0^1 t^{aq-1}(1-t^q)^{b-1} dt = \frac{1}{q}B(a, b),$$

obtained from (1.1) by the change of variables $x = t^q$.

- The choice $a = 1 + p/q$ and $b = 1 - p/q$ produces

$$(6.22) \quad \int_0^1 t^{p+q-1}(1-t^q)^{-p/q} dt = \frac{1}{q}B\left(1 + \frac{p}{q}, 1 - \frac{p}{q}\right) = \frac{p\pi}{q^2} \operatorname{cosec}\left(\frac{p\pi}{q}\right).$$

This appears as **3.251.8**.

- The choice $a = 1/p$ and $b = 1 - 1/p$ gives

$$(6.23) \quad \int_0^1 x^{q/p-1}(1-x^q)^{-1/p} dx = \frac{1}{q}B\left(\frac{1}{p}, 1 - \frac{1}{p}\right) = \frac{\pi}{q} \operatorname{cosec}\left(\frac{\pi}{p}\right).$$

This appears as **3.251.9**.

- The reader can now check that the choice $a = p/q$ and $b = 1 - p/q$ yields the evaluation

$$(6.24) \quad \int_0^1 x^{p-1}(1-x^q)^{-p/q} dx = \frac{1}{q}B\left(\frac{p}{q}, 1 - \frac{p}{q}\right) = \frac{\pi}{q} \operatorname{cosec}\left(\frac{p\pi}{q}\right).$$

This appears as **3.251.10**.

- Putting $v = 1$ and $b = \nu - a$ in (6.1) we get

$$(6.25) \quad \int_0^\infty \frac{t^{ac-1} dt}{(1+ut^c)^\nu} = \frac{1}{cu^a}B(a, \nu - a).$$

Now let $a = r/c$ to obtain

$$(6.26) \quad \int_0^\infty \frac{t^{r-1} dt}{(1+ut^c)^\nu} = \frac{1}{cu^{r/c}}B\left(\frac{r}{c}, \nu - \frac{r}{c}\right).$$

This appears as **3.251.11**.

- We now choose $b = 1 - 1/q$ in (6.21) to obtain

$$(6.27) \quad \int_0^1 \frac{t^{aq-1} dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(a, 1 - \frac{1}{q}\right).$$

Finally, writing $a = c - (m-1)/q$ gives the form

$$(6.28) \quad \int_0^1 \frac{t^{cq-m} dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(c + \frac{1}{q} - \frac{m}{q}, 1 - \frac{1}{q}\right).$$

The special case $q = 2$ produces

$$(6.29) \quad \int_0^1 \frac{t^{2c-m} dt}{\sqrt{1-t^2}} = \frac{1}{2}B\left(c + \frac{1}{2} - \frac{m}{2}, \frac{1}{2}\right) = \frac{\Gamma(c + \frac{1}{2} - \frac{m}{2})\sqrt{\pi}}{2\Gamma(c + 1 - \frac{m}{2})}.$$

In particular, if $c = n + 1$ and $m = 1$ we obtain **3.248.2**:

$$(6.30) \quad \int_0^1 \frac{t^{2n+1} dt}{\sqrt{1-t^2}} = \frac{\sqrt{\pi} n!}{2\Gamma(n+3/2)} = \frac{2^{2n} n!^2}{(2n+1)!}.$$

Similarly, $c = n$ and $m = 0$ yield **3.248.3**:

$$(6.31) \quad \int_0^1 \frac{t^{2n} dt}{\sqrt{1-t^2}} = \frac{\pi}{2^{2n+1}} \frac{(2n)!}{n!^2} = \frac{\pi}{2^{2n+1}} \binom{2n}{n}.$$

In the case $q = 3$ we get

$$(6.32) \quad \int_0^1 \frac{t^{3c-m} dt}{\sqrt[3]{1-t^3}} = \frac{1}{3} B\left(c + \frac{1}{3} - \frac{m}{3}, 1 - \frac{1}{3}\right).$$

This includes **3.267.1** and **3.267.2** in [2]:

$$\begin{aligned} \int_0^1 \frac{t^{3n} dt}{\sqrt[3]{1-t^3}} &= \frac{2\pi}{3\sqrt{3}} \frac{\Gamma(n + \frac{1}{3})}{\Gamma(\frac{1}{3}) \Gamma(n+1)} \\ \int_0^1 \frac{t^{3n-1} dt}{\sqrt[3]{1-t^3}} &= \frac{(n-1)! \Gamma(\frac{2}{3})}{3\Gamma(n + \frac{2}{3})} \end{aligned}$$

The latest edition of [2] has added our suggestion

$$(6.33) \quad \int_0^1 \frac{t^{3n-2} dt}{\sqrt[3]{1-t^3}} = \frac{\Gamma(n - \frac{1}{3}) \Gamma(\frac{2}{3})}{3\Gamma(n + \frac{1}{3})}$$

as **3.267.3**.

7. THE EXPONENTIAL SCALE

We now present examples of (1.1) written in terms of the exponential function. The change of variables $x = e^{-ct}$ in (1.1) yields

$$(7.1) \quad \int_0^\infty e^{-at} (1 - e^{-ct})^{b-1} dt = \frac{1}{c} B\left(\frac{a}{c}, b\right).$$

This appears as **3.312.1** in [2]. On the other hand, if we let $x = e^{-ct}$ in (4.1) we get

$$(7.2) \quad \int_{-\infty}^\infty \frac{e^{-act} dt}{(1 + e^{-ct})^{a+b}} = \frac{1}{c} B(a, b).$$

This appears as **3.313.2** in [2]. The reader can now use the techniques described above to verify

$$(7.3) \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(e^{b/a} + e^{-x/a})^\nu} = a \exp\left[b\left(\mu - \frac{\nu}{a}\right)\right] B(a\mu, \nu - a\mu),$$

that appears as **3.314**. The choice $b = 0$, $\nu = 1$ and relabelling parameters by $a = 1/q$ and $\mu = p$ yields **3.311.3**:

$$(7.4) \quad \int_{-\infty}^\infty \frac{e^{-px} dx}{1 + e^{-qx}} = \frac{1}{q} B\left(\frac{p}{q}, 1 - \frac{p}{q}\right) = \frac{\pi}{q} \operatorname{cosec}\left(\frac{\pi p}{q}\right),$$

using the identity $B(x, 1-x) = \pi \operatorname{cosec}(\pi x)$ in the last step. This is the form given in the table.

The integral **3.311.9**:

$$(7.5) \quad \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{b + e^{-x}} = \pi b^{\mu-1} \operatorname{cosec}(\mu\pi)$$

can be evaluated via the change of variables $t = e^{-x}/b$ and (4.2) to produce

$$(7.6) \quad I = b^{\mu-1} \int_0^{\infty} \frac{t^{\mu-1} dt}{1+t}.$$

8. SOME LOGARITHMIC EXAMPLES

The beta function appears in the evaluation of definite integrals involving logarithms. For example, **4.273** states that

$$(8.1) \quad \int_u^v \left(\ln \frac{x}{u}\right)^{p-1} \left(\ln \frac{v}{x}\right)^{q-1} \frac{dx}{x} = B(p, q) \left(\ln \frac{v}{u}\right)^{p+q-1}.$$

The evaluation is simple: the change of variables $x = ut$ produces, with $c = v/u$,

$$(8.2) \quad I = \int_1^c \ln^{p-1} t (\ln c - \ln t)^{q-1} \frac{dt}{t}.$$

The change of variables $z = \frac{\ln t}{\ln c}$ give the result.

A second example is **4.275.1**:

$$(8.3) \quad \int_0^1 [(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1}] dx = \frac{\Gamma(q)}{\Gamma(p+q)} [\Gamma(p+q) - \Gamma(p)],$$

that should be written as

$$(8.4) \quad \int_0^1 [(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1}] dx = \Gamma(q) - B(p, q).$$

The evaluation is elementary, using Euler form of the gamma function

$$(8.5) \quad \Gamma(q) = \int_0^1 (-\ln x)^{q-1} dx.$$

9. EXAMPLES WITH A FAKE PARAMETER

The evaluation **3.217**:

$$(9.1) \quad \int_0^{\infty} \left(\frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1} x^p} \right) dx = \pi \cot \pi p$$

has the obvious parameter b . We say that this is a *fake parameter* in the sense that a simple scaling shows that the integral is independent of it. Indeed, the change of variables $t = bx$ shows this independence. Therefore the evaluation amounts to showing that

$$(9.2) \quad \int_0^{\infty} \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) dt = \pi \cot \pi p.$$

To achieve this, we let $y = 1/t$ in the second integral to produce

$$(9.3) \quad \lim_{\epsilon \rightarrow 0} \int_0^{\infty} \frac{t^{p-1-\epsilon} dt}{(1+t)^p} - \int_0^{\infty} \frac{t^{\epsilon-1} dt}{(1+t)^{1-p}}.$$

The integrals above evaluate to $B(p - \epsilon, \epsilon) - B(\epsilon, 1 - p - \epsilon)$. Using

$$(9.4) \quad B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

this reduces to

$$(9.5) \quad I = \lim_{\epsilon \rightarrow 0} \epsilon \Gamma(\epsilon) \left(\frac{\Gamma(p - \epsilon)\Gamma(p + \epsilon) \sin(\pi(p + \epsilon)) - \Gamma^2(p) \sin(\pi p)}{\epsilon \Gamma(p)\Gamma(p + \epsilon) \sin(\pi(p + \epsilon))} \right).$$

Now recall that

$$(9.6) \quad \lim_{\epsilon \rightarrow 0} \epsilon \Gamma(\epsilon) = 1$$

and reduce the previous limit to

$$(9.7) \quad I = \frac{1}{\Gamma^2(p) \sin(\pi p)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\Gamma(p - \epsilon)\Gamma(p + \epsilon) \sin(\pi(p + \epsilon)) - \Gamma^2(p) \sin(\pi p)).$$

Using L'Hopital's rule we find that $I = \pi \cot(\pi p)$ as required.

The example **3.218**

$$(9.8) \quad \int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p x^p} dx = \pi \cot \pi p$$

also shows a fake parameter. The change of variable $x = at$ reduces the integral above to

$$(9.9) \quad \int_0^\infty \frac{t^{2p-1} - (1+t)^{2p-1}}{(1+t)^p t^p} dt = \pi \cot \pi p$$

This can be written as

$$(9.10) \quad I = \int_0^\infty \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) dt.$$

The result now follows from (9.2).

10. ANOTHER TYPE OF LOGARITHMIC INTEGRAL

Entry **4.251.1** is

$$(10.1) \quad \int_0^\infty \frac{x^{a-1} \ln x}{x+b} dx = \frac{\pi b^{a-1}}{\sin \pi a} (\ln b - \pi \cot \pi a).$$

To check this evaluation we first scale by $x = bt$ and obtain

$$(10.2) \quad \int_0^\infty \frac{x^{a-1} \ln x}{x+b} dx = b^{a-1} \ln b \int_0^\infty \frac{t^{a-1} dt}{1+t} + b^{a-1} \int_0^\infty \frac{t^{a-1} \ln t}{1+t} dt.$$

The first integral is simply

$$(10.3) \quad \int_0^\infty \frac{t^{a-1} dt}{1+t} = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}.$$

The second one is evaluated as

$$(10.4) \quad \int_0^\infty \frac{t^{a-1} \ln t}{1+t} dt = -\pi^2 \frac{\cos \pi a}{\sin^2(\pi a)}$$

by differentiating (4.1) with respect to a . The evaluation follows from here.

11. A HYPERBOLIC LOOKING INTEGRAL

The evaluation of **3.457.3**:

$$(11.1) \quad \int_{-\infty}^{\infty} \frac{x dx}{(a^2 e^x + e^{-x})^\mu} = -\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a,$$

is done as follows: write

$$(11.2) \quad I = \frac{1}{a^\mu} \int_{-\infty}^{\infty} \frac{x dx}{(ae^x + a^{-1}e^{-x})^\mu}$$

and let $t = ae^x$ to produce

$$(11.3) \quad I = \frac{1}{a^\mu} \int_0^\infty \frac{t^{\mu-1} (\ln t - \ln a) dt}{(1+t^2)^\mu}.$$

The change of variables $s = t^2$ yields

$$(11.4) \quad I = \frac{1}{4a^\mu} \int_0^\infty \frac{s^{\mu/2-1} \ln s ds}{(1+s)^\mu} - \frac{\ln a}{2a^\mu} \int_0^\infty \frac{s^{\mu/2-1} ds}{(1+s)^\mu}.$$

The first integral vanishes. This follows directly from the change $s \mapsto 1/s$. The second integral is the beta value indicated in the formula.

In particular, the value $a = 1$ yields

$$(11.5) \quad \int_{-\infty}^{\infty} \frac{x dx}{\cosh^\mu x} = 0.$$

Differentiating with respect to μ produces

$$(11.6) \quad \int_{-\infty}^{\infty} x \ln \cosh x dx = 0,$$

that appears as **4.321.1** in [2].

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